Suppose a company can always make a profit of 10% on volume; furthermore, that volume will always be ten times net worth. If all the profit is plowed back into growth, what will the growth be? No problem; even an MBA should be able to figure out the volume will increase exponentially, doubling every 8 1/3 months. But what if there are three competing companies, whose total volume is constant, and whose profit/growth feedbacks are more complex?

Almost any really useful problem will have complications beyond the range of "high-class" mathematics. The answer? Put on your blue collar and use system dynamics; let your computer finite-difference its way through your problem. But wait! Components of your problem -- maybe a Markov process, a queuing solution, or some solvable differential equation -- are analytically tractable. Back comes the white collar. Worse yet, system dynamics itself turns out to have been invaded by academics, and some of their contributions can be ignored only at your peril.

This book covers the rudiments of system dynamics and associated modeling. It has grown out of a set of notes for a course at the George Washington University. ORSA hopes that this publication will serve not only as a self-study resource for O/R practitioners, but also as a stimulant to the other faculties who want to provide their students with a background in the subject.

By the way, if it weren't for this remark, we would have left it entirely to the reader to pose and solve the question of why a fractal is on the cover of a book on system dynamics.

John D. Kettelle
Editor-in-Chief
ORSA Softback Books
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PREFACE

A systems approach to problem solving requires having the perspective to deduce important system variables, and the relationships between them. Understanding then evolves from modeling the system, and seeing how the variables change, over time, as the problem parameters are changed. System Dynamics provides a method for developing such understanding. This monograph is a "one weekend" introduction to system dynamics.

It is intended for both the academic student and the manager or policy maker. In an academic setting the material would comprise part of a college level course on quantitative methods. The typical reader would be interested in the management sciences, and would have some proficiency in mathematics, though the discussion is largely nonmathematical. Familiarity with algebra is assumed. Calculus is useful but not essential.

The presentation of material is provided in a sequence that begins with a general introduction, with material then repeated and refined in later chapters. Chapter one's concepts are amplified in chapters two and three, but may not be clear until the mechanics of model building and equation writing in chapter four are completed. The reader with a systems background may want to begin with chapter 4. Chapter five discusses functions, and emphasizes the value of disaggregating systems into understandable components. Chapter six provides sample problems, with solutions given in the appendix. It will prove useful to browse through the appendix even before beginning chapter one, to see the problem/solution format of the method.

System Dynamics has developed under Jay Forrester of MIT, and been enhanced by scholars worldwide. Dennis Meadows, Alexander Pugh, Barry Richmond, George Richardson, Geoffrey Coyle, Henry Weil, Jean-Claude Huot, and Ray Shreckengost directly influenced me. Kenan Sahin introduced me to the subject. I draw from their work, but attribute no errors to them. Jane Marden spent many hours arranging the work into its present form.

This introduction evolved from courses offered at The George Washington University. I am grateful to its Department of Operations Research for incorporating this material into its management science concentration, and to the students there who tolerated its development.

Rolf Clark
## Systems Modeling and System Dynamics: A Tutorial Guide

### Contents

<table>
<thead>
<tr>
<th>Chapter and Section:</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Chapter 1: ORIENTATION</strong></td>
<td></td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Structure</td>
<td>2</td>
</tr>
<tr>
<td>1.3 General Observations on Systems</td>
<td>5</td>
</tr>
<tr>
<td>1.4 On Models and Modeling</td>
<td>7</td>
</tr>
<tr>
<td>1.5 On Understanding and Control</td>
<td>8</td>
</tr>
<tr>
<td>1.6 On Solutions and Exploration</td>
<td>9</td>
</tr>
<tr>
<td><strong>Chapter 2: SYSTEMS AND SYSTEM CONCEPTS</strong></td>
<td></td>
</tr>
<tr>
<td>2.1 Systems and System Concepts</td>
<td>11</td>
</tr>
<tr>
<td>2.2 The &quot;Time&quot; Variable</td>
<td>13</td>
</tr>
<tr>
<td>2.3 Dynamic Systems -- The Process of Accumulation</td>
<td>14</td>
</tr>
<tr>
<td>2.4 Feedback Systems</td>
<td>16</td>
</tr>
<tr>
<td>Exercise 2.4.1</td>
<td>19</td>
</tr>
<tr>
<td>Answer 2.4.1</td>
<td>20</td>
</tr>
<tr>
<td>2.5 Conservative Flows and Information Flows</td>
<td>25</td>
</tr>
<tr>
<td>2.6 Variables, Parameters, and System Boundaries</td>
<td>29</td>
</tr>
<tr>
<td>2.7 The System Boundary</td>
<td>32</td>
</tr>
<tr>
<td>2.8 Model Sectors</td>
<td>33</td>
</tr>
<tr>
<td><strong>Chapter 3: ANALYZING A SYSTEM THROUGH INTEGRATION</strong></td>
<td></td>
</tr>
<tr>
<td>3.1 Functions and Integrals of Functions</td>
<td>36</td>
</tr>
<tr>
<td>3.2 Disaggregating Equations</td>
<td>39</td>
</tr>
<tr>
<td>3.3 How Disaggregating a System Simplifies System Analysis</td>
<td>42</td>
</tr>
<tr>
<td>3.4 Iteration</td>
<td>46</td>
</tr>
<tr>
<td>3.5 System States</td>
<td>50</td>
</tr>
<tr>
<td>3.6 Lags or Delays</td>
<td>52</td>
</tr>
<tr>
<td><strong>Chapter 4: MODELING, THE MECHANICS</strong></td>
<td></td>
</tr>
<tr>
<td>4.1 The Time Step and its Notation</td>
<td>54</td>
</tr>
<tr>
<td>4.2 Level and Rates, the Conservative Components</td>
<td>56</td>
</tr>
<tr>
<td>4.3 Auxiliaries and Constants</td>
<td>59</td>
</tr>
<tr>
<td>4.4 Initial Values, Model Specifications, and Labels</td>
<td>62</td>
</tr>
<tr>
<td>4.5 Logic and Test Functions</td>
<td>65</td>
</tr>
<tr>
<td>4.6 Delays in DYNAMO</td>
<td>67</td>
</tr>
</tbody>
</table>
SYSTEMS MODELING AND SYSTEM DYNAMICS:
A TUTORIAL GUIDE

1. ORIENTATION

System Dynamics\(^1\) is a method for analyzing system problems. It is based on a straightforward "stocks and flows" structure, designed for modeling systems with numerous variables, and with lagged feedbacks between variables. Many managerial systems and most social systems have such feedbacks. In their transient states, such systems are virtually impossible to solve mathematically, so they are usually simulated. System dynamics simulates such systems on microcomputers, using inexpensive software that now includes graphic output, model documentation, and report generation. This guide provides an introduction to systems, and modeling them using system dynamics.

System dynamics is useful in policy analysis. The following highly simplified planning problem for the U. S. Navy represents a typical policy problem:

"Three percent real budget growth is anticipated over the next ten years, and acquisitions must be traded off against ownership costs (manning, maintenance, and operations) in each year's planned budget. Increasing acquisition of ships and aircraft means that ownership funds must be reduced, leading to readiness decay. Increasing ownership funding, on the other hand, means less acquisitions and reduced future force levels, yet the enhanced readiness may compensate for the smaller forces. How should the acquisition/ownership tradeoffs

\(^1\) Originally introduced as "Industrial Dynamics" by MIT's Jay W. Forrester [Forrester, 1961].
be appraised, and how should budgets be adjusted when an unexpected cut in a budget occurs?"

In this problem a financial system allocates the scarce resources of an organization acquiring and operating weapon assets. It is typical of the type of problem we would like to explore using system dynamics.

1.1 Introduction

Most realistic problems involve systems and the way they change. We will discuss using the method of system dynamics for thinking about, understanding, and modeling processes so that problems can be explored and decisions made.

A changing system has two descriptive qualities. The first concerns the materials flowing through the system. The second is the configuration (or "structure") that contains and controls the flow of materials. Miser and Quade (1985) open their text on "Systems Analysis" with an appropriate definition:

"Many of society's problems emerge from processes associated with structures that combine people and the natural environment with various artifacts of man and his technology; these structures can be thought of as systems."

A structure can exist without materials flowing through it, but we shall normally include the materials flowing within the structure as part of the system.

The manufacture of automobiles has material components such as raw materials and a production labor force, contained in a structure that includes factories, distribution systems,
managerial controls, financial policies, and so forth. The complex process of manufacturing cars occurs within a system that includes production, distribution, and marketing sectors.

A bathtub filling up with water is a more simple process. Its only material flow is water. The bath water flows within a structure composed of pipes and the bathtub itself. The pipes and tub, with the water, make up the system. If a human controls the water flow, then the human is part of the system.

Processes are modeled by defining the structure and its control variables and then letting the material flows occur. This requires defining certain components of the system structure, the rules that cause the components to control the material flows, and the status of the material flows at the start time ("time zero"), and then calculating the values of the system variables as time passes.

"Materials" should be interpreted in the broad sense as anything that can be accumulated. Automobiles and water are materials, but, in systems theory, so are humans. More abstractly, a human quality, such as "worker productivity" can be material, for it accumulates as work accomplished. Other human constructs, such as acceleration (which accumulates as velocity), price change (which accumulates as price level), and attitude change (which might accumulate as a community's political position) can all be considered "materials" flowing through a structure, even though some of the materials seem abstract.
System Dynamics is the study of processes through the use of systems and how they can be modeled, explored, and explained. System modeling can be an elusive skill, and a discussion of modeling seems appropriate.

A model is the representation of a real process. Models allow studying the process in a laboratory setting -- our lab will be a micro-computer and its predecessor, paper and pencil. We want to be able to build models on a computer screen, or on a desk pad.

It is very different to build a model, than it is to use a model that has been built. By modeling, we will mean model building.

To "model" something requires being able to structure one's thinking about the system being studied. It means defining a model that works almost like the real system. That is, when materials flow within the modeled system, the various flows and accumulations should provide insight into the real process under study.

Modeling requires both original thinking and practice. It requires developing structure from non-structure, sometimes from chaos. Transforming chaos into structure is aided if one understands systems, and how they control the materials flowing within them.

System dynamics provides definitions, bookkeeping tools, and communication methods for system modeling. It is not so much a
"technique", as it is a set of analytic tools for the modeler to use in structuring a problem.

1.2 Structure

It is not always clear how to begin to model a system. "Structure" provides the framework for beginning. Seeing a house being constructed, the foundation poured and walls framed with 2x4 boards, provides a sense of how a house is structured. Without ever seeing the building process, one might not know where to begin.

System dynamics provides a structure for modeling systems. Let us begin to develop that structure by first considering a sample problem, and how it may be viewed.

The reader is asked to consider how one might model the process of a proprietor pricing tire gauges, one of the products sold in his hardware store. How would you proceed? What are the variables of this problem, and how they are related. Feel free to scribble a few notes on the space that follows.
One might try to apply microeconomic's supply and demand curves to determine equilibrium price. It does not help in the proprietor's case. Supply and demand theory is a "model" in the sense that it predicts where equilibrium price will be in steady state, that is, when the system of buying and selling tire gauges has stabilized. The theory requires knowing supply and demand curves.

But these curves, along with the equilibrium condition that "supply equals demand", are the results of many proprietors' actions on pricing, not the causes for this proprietor to change prices. We should use causes, not results, to predict actions. Telling the proprietor to equate supply and demand to establish the price level would be like telling a golfer to hit the ball straight and far to establish a good swing.

What really happens in the hardware store is that the proprietor regularly checks inventory and compares it to a desired inventory level. The desired level probably depends on recent sales, a good indicator of future sales to be filled from the inventory. The magnitude of the difference between actual inventory and desired inventory causes the proprietor to change prices. Price changes feedback to effect sales, which in turn affect both inventory and desired inventory to change again. The variables are interrelated. This dynamic, feedback problem is complex, yet we shall soon be able to model it quite easily.

The logic represents a causal modeling approach to the proprietor's problem. It defines the underlying relationships
producing actions and their impacts. Still, building a model of the process requires a structure, that is a set of equations that allow simulating the inventory and pricing dynamics. System Dynamics provides that structure based on the concept of stocks and flows.

Stocks are accumulations of flows -- an inventory is a stock accumulating the ordered goods flowing into the inventory, less those flowing out as sales. Price is another stock, and changes to price are flows -- they are the rates of change of price, and are added up, or accumulated, to provide the current price. The stocks of inventories and prices, their associated flows, and how they effect each other, would allow us to "model" the proprietor's pricing problem. We will develop stock/flow models of this type as we proceed.

1.3 General Observations on Systems

Systems surround us. Crime in the streets, the national deficit, a runner's physiology, the clock's pendulum, a lake filling with water. All are systems.

The most interesting systems are those in a state of change -- in a "transient" state. Systems that have stabilized are less interesting -- analytically at least. Once they have reached a steady state, systems do not require further analysis or even observation, for they will not change.

Some systems can be merely observed, while others need to be controlled. Early man, and settlers in new lands, if surrounded
by abundance, lived in environmental systems that did not need to be controlled. Control becomes more important as an environment becomes crowded, and its resources scarce. The early settlers in Florida hardly needed to worry about the water supply for irrigation. Today, controlling Florida's water supply is a major policy issue.

As policy makers need better ways of observing and controlling systems, the need for system modeling to test policies becomes more important.

The ability to model systems not yet studied becomes a valuable skill as the world becomes complex and new problems arise. The management scientist will need to incorporate, more and more, the simulation of complex systems in their transient states.

1.4 On Models and Modeling

It is typically easier to use something than to design or build it. Many individuals can drive a car. Few could design one, especially if it had not been done before. The same is true of analytic models -- it is easier to apply one already designed to analyze a system, than to design and build one from scratch for a system not yet analyzed.

Modeling requires the ability to think and to structure, abilities that are not easily acquired. Modeling is an art, and while not everyone can be a successful artist, there are certain concepts of art that can be learned. Conceptualization, layout,
color coordination, perspective, and the use of light and shadows, for example, can be studied by the prospective artist. Different mediums can be learned. The works of successful painters can be seen and evaluated, and selected objects and scenes can be drawn for practice. Such knowledge helps one become an artist.

So it is with systems modeling. System components and structure, system states and boundaries, and the use of variables and parameters can be studied. Models previously developed can be studied, and simple systems can be modeled for practice. Available software can be learned.

The art of modeling will be facilitated if the principles of systems are understood. Fortunately, the fundamentals of systems are limited. They include system components (levels, rates, and information "auxiliaries"), system states (initial, transient, and "steady state"), and system definition (the functional forms of equations defining the system). They also include concepts such as accumulations, rates of change, delays, and feedbacks.

The remaining chapters deal with these concepts.

1.5 On Understanding and Control

Control requires understanding, and understanding is aided by the ability to quantitatively model a system. The computer and modern software have made it possible to model and study systems, and therefore to begin to understand and control them.
Controlling means more than being able to implement changes. In most managerial settings, control also requires observing, understanding, and communicating the need for, and effects of, policy changes, as policy makers must be convinced that change is needed before it will be authorized.

System dynamics is not mathematically complex, and is understandable to most audiences. Mathematics is, after all, only one language to describe phenomena. The computer, with the aid of modern simulation software and methods, provides another language. Its graphical outputs are easily absorbed by managers and policy makers.

Deemphasizing mathematical exposition does not mean sacrificing systems that can be studied and controlled. Quite the opposite. Far more complex, dynamic, ill-behaved systems can be modeled using simulation, than can be analytically treated.

The systems that can be analyzed and controlled mathematically tend to be well defined, precise physical systems controlled by physical laws. Orbitting satellites and swinging pendulums are such systems. Most human problems are neither well defined nor well behaved, and are affected by human interactions and inconsistencies. They therefore defy mathematical analysis. They can only be understood and possibly controlled through more comprehensive, or perhaps more flexible approaches. Simulation becomes approriate for such problems.
1.6 On Solutions and Exploration

A final orientation issue involves the difference between solving and exploring. Analytic (that is, mathematical) solutions to systems usually require satisfying an objective while controlling the system variables. This typically means having an objective maximized or minimized while keeping the system within allowed boundaries. Or, for a system in steady state, an analytic solution means solving a set of equations simultaneously.

Exploring, on the other hand, means asking "what-ifs", by changing the model values and then simulating the system again to see the impact of the changes. There may be an objective, but while it can seldom be optimized, it can usually be explored. "Exploring" allows seeing how the objectives change when the control variables are altered. But while most problems can be explored, a disadvantage of simulation is that the problems cannot generally be "solved.

Further, only a limited set of explorations can be performed -- even with modern computers, there are still limits to the number of variables and time periods that can be investigated. If 10 variables each have 10 possible values in each of 10 time periods, there will be more possible combinations to explore than there are atoms in the earth. Fortunately, most of the combinations will not be feasible, and a logical direction of exploration will usually be apparent in the search, but seldom will the optimal path over the time horizon of interest be found.
In brief, while mathematical approaches allow finding optimal solutions, system simulations only find a solution better than the existing one -- but not always the best solution. On the other hand simulation allows studying a very large number of realistic systems, most of which cannot be solved analytically, at least not in their transient states. Simulating a process--such as a social or economic system -- from its present state through a logical sequence of follow on states resulting from a selected policy, allows seeing the results of that policy throughout the time spectrum of interest. That may determine whether a new policy should be effected or not. Some of our legislative policies might benefit if analyzed that way.
2. SYSTEMS AND SYSTEM CONCEPTS

This chapter discusses important aspects of a system in general terms.

2.1 Systems in General

A system is a collection of components, interacting for a purpose.

The key words in this definition -- general enough to cover many problems -- are "components" and "interacting". Yet a pair of pliers has components that interact, and is therefore a system, but not one to attract our current attention. The definition is too broad to be useful. It is better to discuss the characteristics of a typical system that might be studied in management science, especially that end of the science concerned with policy analysis.

Systems of interest have something flowing through them, and the flows accumulate in various parts of the system. The flows of a fuel distribution system occur within pipelines, and the flows are accumulated in fuel tanks. An inventory system, an influenza epidemic, an irrigation system, all have analogous flows and accumulations. A pair of pliers does not.

The "flows" are typically flows of materials that accumulate as "stocks" -- the flows of goods ordered and goods sold, for example, accumulate as a stock held in a warehouse. Material
flows should be interpreted in the broadest generic sense. "Material" may be people, or energy, or temperature, or other abstract concepts that can be accumulated. Changes to temperature, for example, may be accumulated as current temperature. Price levels are accumulations of price changes. Modeling will require considerable imagination on what are and are not likely candidates to be material flows in any particular system.

A system encompasses three major concepts. The first is time, the second accumulation, and the third feedback.

Consider a simple inventory system. The inventory changes over time. The change is due to the accumulation of orders. And the order rate changes due to the feedback effect of inventories exceeding or falling short of desired inventory levels.

As time, accumulation, and feedback are so basic to systems, they warrant further discussion.

2.2 The "Time" Variable

By analyzing a system, we mean showing how the system changes over time. This can be done in two ways. First by finding an analytic function that includes all the system variables in a mathematical expression that is a function of time, and then inserting any selected value of time to obtain system values at that point. This analytic approach is virtually impossible in any realistic managerial or social system. There are too many interacting variables to allow solution.
The second method is to simulate the system. This means providing a set of equations that defines each variable in the system as a function of other variables which affect it. Then, given a set of initial conditions -- values for appropriate variables at the start of the simulation -- the system is "simulated" by calculating new values for each variable at each successive time step. Time is the indexing variable. The system variables are calculated as time changes, and the values of the variables are indexed, or subscripted, by the point in time at which they are calculated.

The second method is usually feasible for use in managerial and social systems.

A very important characteristic of time is that it is a truly independent variable (relativity theory aside). It makes sense to make the level of an inventory be a function of time, but it hardly seems practical to make time a function of inventory level. Being an independent (or exogenous) variable, time is an excellent indexing variable. Systems with feedback can be made dependent on time, without concern that time is dependent on the system variables.

There are modeling techniques that use other indexing variables. An "event-based" simulation of an inventory, for example, may define its variables as functions of sales, so that
a calculation occurs each time an item is sold. Such indexing is not very suitable however, for most feedback models.¹

2.3 Dynamic Systems -- The Process of Accumulation

A dynamic system is one in which there are flows that, as time passes, are accumulated (or "integrated") into stocks. Flows are often called "rates", for "rate of change" per unit time, of the stock, and stocks are called "levels", representing the amount of flows that have accumulated. The terms "stocks and flows" and "levels and rates" are closely related. A level represents the value for a stock at a point in time, and a rate represents the value for a flow.

Whenever flows -- which can be inflows or outflows -- are being accumulated or integrated into a level, then the system controlling the process is a dynamic system, and the system is in a dynamic state if one or more of the levels in the system are changing, that is the net accumulation rates are not all permanently zero in all the levels. If the net accumulations rates are stable at zero in all levels, the system is in a static state.

For example, consider an irrigation system consisting of two dams placed along a river basin to accumulate two reservoirs (or stocks) of water. During any period of time, each reservoir

¹ Unless the new indexing variable is also independent of other problem variables. It is very difficult to define the other variables, while the system is in its transient state, if they are dependent on the indexing variable, and the indexing variable is dependent on them, that is, there is feedback.
accumulates the net of its inflow less outflow from its dam. If the reservoir levels of each dam are unchanging, then the outflows equal the inflows and this system is static, there are no net accumulations in any of the dams.

Now assume a thunderstorm occurs in the high country feeding the tributaries above the dam furthest upstream. After a lag, this first reservoir will begin accumulating the rainfall. Its inflow will exceed its outflow and net accumulation for the first dam becomes positive. The system becomes dynamic -- even though the second dam has not yet felt the effect of the rainfall. So long as one level in a system is in a dynamic state, the system--by definition -- is in a dynamic state. This is also called being in a "transient state".

As the thunderstorm subsides, the first dam will eventually return to a static state, where inflow and outflow are equal, while the second dam absorbs the effects of the storm. Ultimately, the dynamic effects of the storm are gone, as each dam returns to a static condition, and the level of each dam is once again constant.

But note two things. First, the new static levels of each dam may be different than the previous static state. Second, "static" does not necessarily mean there are no accumulations. Each dam can still accumulate inflow, but the inflow now equals the outflow. A static system can have levels, so long as they
are unchanging. A system with levels that have stabilized, or become constant, is said to be in "steady state".\textsuperscript{2}

There are systems that have no levels (accumulated flows) at all. In the river system just mentioned, the river itself, before the dams were built, could be considered a system without accumulations. Ignoring friction -- and any accumulated water in natural pools along the river -- the purely static river system would simply be a river which flowed either slowly or rapidly, but had no changing stocks or levels.

A more precise version of such a system is a lawn sprinkler system, which has pipes from the water supply to the sprinkler heads. In this system -- static by our definition -- water flows through the pipes, which always contain the same volume and no accumulations.

Often, in "systems analysis" or other forms of management science, a system is analyzed, and results are presented for the system only in its steady state. This allows solution through static analytic methods, mathematically tractable and academically satisfying, but somewhat unrealistic.

To be specific, if an inherently dynamic system is assumed to have reached steady state, then the analysis of the system simply requires making the inflows to each level equal its outflows. Equalizing these flow rates provides an "equilibrium condition" for each level, and the condition is algebraic ...  

\textsuperscript{2} The definition needs to be modified if randomness is introduced. Steady state then means a constant \textit{average} value for a level has been reached.
that is, there are no derivatives or integrals involved in the equation. Simultaneously considering the individual equilibrium conditions for each level allows solving a set of simultaneous algebraic equations -- instead of solving the inherent differential equations needed to define the transient state. Said another way, algebra can be used to solve the system equations, instead of calculus. This is convenient mathematically, but avoids exploration of the transient dynamics that accompany most systems.

But policy analysts need no longer emphasize steady state analysis. Modern computers and software have made it feasible to explore dynamic systems. After all, knowing that the nation's unemployment rate will reach four percent in steady state, is not comforting if the rate fluctuates between 12 percent, four, eight, five, and so on, before stabilizing at four percent ... especially since policy and economic changes will no doubt prevent the steady state from ever occurring!

2.4 Feedback Systems

A system can have accumulations -- be dynamic -- without having feedback. Systems that vary over time without feedbacks have been referred to as being "dynamic in the weak sense", while those having feedback have been called "dynamic in the strong sense" [Brems, 1973]. The difference lies in the concept of feedback.
Feedback, for our purposes, means that at least one variable in a system is affected by another variable, which was earlier affected by the first. In other words, there is a "feedback loop" somewhere in the system.³

For example, the rate of new births in a population is affected by the population size, and of course the population size depends on the rate of new births. That is feedback by our definition.

The feedback characteristic is the ingredient that complicates the mathematical solution of a system. The following simple exercise sequence is intended to show the increase in conceptual difficulty that occurs in going from a static system, to a dynamic system, to a dynamic feedback system.

Exercise 2.4.1 A Water Flow Problem

The following three exercises demonstrate, respectively, static, dynamic, and feedback conditions. The problem uses several concepts not yet fully developed. The reader new to systems theory may want to follow the problem logic without becoming overly concerned if, for now, portions of it seem still undefined.

³ In control system theory, a feedback loop is a subsystem in which the difference between a desired value and actual value of a controlled variable is used to alter the input to the subsystem so that the actual variable is driven toward the desired value. (Bateson, 1980: 10-13).
E2.4.1A: Water flows into a tank at a rate inversely proportional to the level in the tank. The inflow is 1 gallon per minute at time zero when the tank is empty, and 0 when the tank contains four gallons. In other words, the inflow rate can be defined mathematically:

\[ \text{INFLOW}(t, t+\Delta t) = 1 \times \frac{(4-\text{LEVEL}(t))}{4}, \]

where * is the symbol for multiplication, \( \Delta t \) is a small increment of time, \( \text{INFLOW}(t, t+\Delta t) \) is the value of the inflow between time \( t \) and \( t+\Delta t \), and where \( \text{LEVEL}(t) \) is the value of the variable LEVEL at time \( t \).

Further, an outflow valve lets water flow out of the tank at a rate equal to 0.2 times the number of gallons in the tank. What is the steady state level of water in the tank, and what does the flow equal at steady state, that is, when the inflow equals the outflow and the level of water in the tank has stabilized? The answers are provided at the end of this section.

E2.4.1B: Water flows into a tank with no outflow. At time zero, the tank is empty and the inflow is 1 gallon per minute. The inflow decreases proportionally each minute with time until time 4, when the flow is zero. In other words, the inflow rate is

\[ \text{INFLOW}(t, t+\Delta t) = 1 \times \frac{(4-\text{TIME}(t))}{4}, \]

and the level at time \( t=T \) is

21
LEVEL(T) = SUM(INFLOW(t\(-\Delta t, t); t=\Delta t, T),

where SUM(INFLOW(t\(-\Delta t, t); t=\Delta t, T) means

\[ \text{INFLOW}(0, \Delta t) + \text{INFLOW}(\Delta t, 2\Delta t) + \ldots + \text{INFLOW}(T-\Delta t, T). \]

What are the values for the level and the rate of flow of water into the tank over time? Use the following graphs to obtain an approximate solution (or solve the problem mathematically using calculus). To simplify the analysis, assume \( \Delta t = 1 \) in the above equations.
Ex. 2.4.1C: Water flows into a tank with no outflow. The inflow is 1 gallon per minute when the tank is empty, and decreases proportionately with the water level until the level equals 4 gallons, when the flow is zero. In other words, the inflow rate is

\[ \text{INFLOW}(t, t+1) = 1 \times (4 - \text{LEVEL}(t))/4, \]

and the level at time \( T \) is
LEVEL(T) = SUM(INFLOW(t-1,t); t=1,T).

What is the flow rate into the tank at time 3, and what is the water level at that time? Use the following graphs to obtain an approximate solution or solve the problem mathematically using calculus.
Answer 2.4.1: Water Flow Problem

A2.4.1A: At steady state, the water inflow equals the outflow. Assume the symbol $\infty$ means "a very large number" and let any variable $\text{FLOW}^\infty$ be designated $\text{FLOW}(\infty)$. Then mathematically,

\[
\text{INFLOW}(\infty) = 1 \times (4 - \text{LEVEL}(\infty))/4 \\
\text{OUTFLOW}(\infty) = 0.2 \times \text{LEVEL}(\infty)
\]

(2.4.1) (2.4.2)

and $\text{INFLOW} = \text{OUTFLOW}$, the equilibrium condition, implies

\[
\text{LEVEL}(\infty) = 2.22 \text{ GALLONS, and} \\
\text{INFLOW}(\infty) = \text{OUTFLOW}(\infty) = 0.444 \text{ gallons/min.}
\]

These are the steady state values obtained by solving the equilibrium condition using 2.4.1 and 2.4.2. This is a static mathematical exercise, as there are no difference equations.

A2.4.1B: This problem has accumulation, but no feedback, the variable INFLOW is dependent on time, and time is always independent, therefore exogenous to the problem. The graphical solution is thus easily derived, and is shown. The system is "iterated" through time by starting with the initial conditions (water level equals zero at time zero), then adding to the level, each one minute time step ($^*t=1$), the amount of water that has flowed into the level since the last iteration. The solid lines of figure A2.4.1B show the results of this iterative process.
Iteration, and the treatment of time and time intervals, will be discussed more fully in section 3.4. For now, note that iterating the solutions at one minute intervals causes inaccuracies. The solid lines in the figure show how flow rates are held constant for a entire time step. The dashed lines show the correct flows and levels, when flows change continuously instead of in discrete time steps. The dashed line solution results from the calculus solution provided at the end of this exercise. Accuracy increases if $\Delta t$ is reduced, but the number of calculations expands. Note also that the level, being an
accumulation, has a value equal to the area under the flow graphed to that point.

A2.4.1C: When the inflow rate depends on the level, instead of on the exogenous variable time, the system has feedback as well as accumulation. Feedback complicates the solution significantly, as the inflow rate affects the water level, and the water level affects the inflow rate. A graphical solution requires first finding the flow rate at each time step, then calculating the level, recalculating the flow for the next time step, then recalculating the level, and so forth.
Digression: Mathematical solutions to exercises E2.4.1B and E2.4.1C provide insight. Using calculus, the rate of flow in exercise B can be written as the change in level with respect to time (the "derivative")

$$\frac{dL}{dt} = 1 \cdot (4-t)/4; \ 0 \leq t \leq 4. \quad (2.4.3)$$

Note that $dL/dt$ is only a function of $t$. The solution to 2.4.3 is (separating variables and using direct integration to solve the differential equation)

$$L(t) = t - t^2/8. \quad (2.4.4)$$

In exercise C, the rate of flow is

$$\frac{dL}{dt} = (4-L(t))/4; \ 0 \leq L \leq 4. \quad (2.4.5)$$

Here $dL/dt$ is a function of both $L$ and $t$, and the problem is more sophisticated. Solving 2.4.5, a very simple linear differential equation, is already non-trivial. The solution method, found in any basic calculus text, leads to

$$L(t) = 4 - 4e^{-t/4} \quad (2.4.6)$$

2.4.4 and 2.4.6 can be checked by the reader to confirm they fit the conditions of the exercise, for example, when $t=0$, $L=0$. 

28
Exercise 2.4.1 has represented a very simple system. The mathematical digression was meant to show how analytically solving a system rapidly becomes difficult as the number of levels in a system increases and the interactions of their rates of flow becomes more complex. Systems with over three or four levels become difficult, mathematically. A social system with 10 or 15 levels simply cannot be solved analytically.

2.5 Conservative Flows and Informational Flows

We have discussed two systems components -- levels and rates (or stocks and flows). Any particle of material being processed in a system can only be in one place at any point in time, and it must either be in a rate flowing between two levels, or be in a level. Rates and levels, therefore, are considered to encompass the "conservative" parts of a system: material flowing from a source is no longer at that source; material transferred from one part of the system to another cannot be in both places; material cannot be transferred to more than one place at a time. This is not so with information.

Information about material in one part of the system can be transferred to several other parts, yet the material still remains at the source. Information flow is not conservative. Information on the level of inventory held at the end of the

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4 Note the implicit assumption that the material in a system is infinitely divisible. It is feasible, though cumbersome, to simulate discrete flows. For most purposes, and particularly for the present discussion, infinitely divisible flows are adequate.
month, for example, can be used by the marketing department to affect the advertising rate, by sales to affect the rate of price changes, and by purchasing to affect the ordering rate. Information taken from a source remains at the source.

In all systems, the rates of flow control the system, and the values of the rates depend on information about the system's levels. This may be clearer if rates are compared with levels. A level changes as, and only as, material flows into or out of the level ... the level depends on rates directly connected to it. Rates depend on levels, but not in an analogous way. A rate can depend on a level not connected to the rate. The rate of population deaths, for example, can be a direct function of pollution, a level defined in a completely different part of the system. However the pollution level could not be directly affected by population deaths.

A key difference, then, is that levels depend on material flows connected to them, while rates depend on information links to levels anywhere in the system. These information links to levels are functional relationships that can become quite complex mathematically, and therefore are often decomposed through the use of intermediate variables, called "auxiliaries". Auxiliaries can be functions of levels, rates, or other auxiliaries. These functions help clarify the definition of rates and their ultimate dependence on the system levels.

The relationship between levels, rates, and auxiliaries, will become clear when they are diagrammed in sections 4.2 and
4.3, but as an example, consider the inventory level for a microcomputer store. Computers are ordered and accumulate in the store's inventory, and are sold. The desired inventory level (a constant) is 100 computers. In this system, the rate of arrival of computers is a flow, the rate of sales is a flow, the accumulated computers in the inventory is a stock, and the information variable "desired inventory minus actual inventory" is an auxiliary that controls orders for new computers. If, over time, average weekly sales of computers (which depends on weekly sales) rises permanently, then the desired inventory level may change to a new value. Desired inventory, if dependent on average sales, ceases being a constant, and becomes another information auxiliary.

The control variable in this problem is the flow of orders placed with the manufacturer, which affects the flow of computers arriving at the store.

Control variables are the rates used by management to control a system. Proper control of the control variables will cause the system to respond properly, while poor control will cause undesired system responses. Managerial control invariably depends on information about the system's status. Consequently control variables respond to information -- information which depends on the status of the system's levels, which, after a time lag, depend on the control variables. That is the feedback nature of systems.
2.6 Variables, Parameters, and System Boundaries

Rates, levels, and auxiliaries have all been called variables. Each of these system components indeed vary as the system changes. These system variables are more correctly called "endogenous variables", as they are determined internal to the system.

Variables that do not depend on other system variables can be referred to as "exogenous variables", as their values are determined outside the system. Time is the most common exogenous variable -- its value changes, but is not dependent on the values of system variables.

The opposite term to "variable" is "parameter". Constants are parameters.

In an inventory problem, the delivery delay between placing an order from the manufacturer and receiving the order in inventory may be a constant, say three days. The delivery delay is a parameter.

The sales rate, on the other hand, may vary only with time of year (being high in the summer and low in the winter), or vary randomly from week to week. In this example, sales rate is an exogenous variable -- it does not depend on other endogenous system variables.

Inventory level, however, depends on deliveries, which is a system variable depending on inventory. Inventory level is an endogenous variable, as, by the way, is the delivery rate. When we refer to system variables, we shall mean endogenous variables.
The key difference between system variables and either parameters or exogenous variables, is that system variables depend on other variables within the system, while parameters and exogenous variables depend on things outside the system.

Parameters and exogenous variables are typically determined through statistical methods such as estimation, regression, and forecasting.

Parameters and exogenous variables then, are the links that connect the variables within a system to the environment outside the system; they cross the system boundary.

2.7 The System Boundary

When studying a system and how the system changes, the modeler must identify the variables of major interest. The boundary of the system being modeled should circumscribe all such variables. System variables are related to exogenous variables and parameters external to the model either through the use of analytic functions, or through "table functions" which are described later.

The boundary is often not explicitly defined in the modeling process. It implicitly contains all variables that are defined as dependent on other variables, and excludes those only dependent on constants or exogenous variables. In figure 2.7.1, recruitment, labor supply, resignation, and labor demand are endogenous variables. Recruitment and resignation depend on
(exogenous) pay scales and unemployment rates, and labor demand depends on the (exogenous) military threat.

![Diagram of model boundary for military labor problem.](image)

**Figure 2.7.1: Model Boundary for Military Labor Problem.**

In figure 2.7.2, the modeler has made pay scales endogenous, feeling that if military labor supply and demand are mismatched, the organization under study can affect pay scales to help correct the mismatch.

These figures introduce influence diagrams typical in model building. The arrows show the direction of causal influence. For example, the larger the labor supply, the larger the resignation rate, and the larger the resignation rate the less the labor supply.
In most model efforts there will be certain variables that are neither clearly endogenous or exogenous. In a military manpower planning process, for example, recruiting rates are dependent on advertising budgets and military pay rates, and all three are likely to be endogenous variables. Recruiting rates are also dependent on the demographics of the nation's 18-21 year olds, and the size of this labor pool can be provided to the model as a time dependent exogenous variable. However, while the pool of all 18-21 year olds is virtually independent of the military recruiting rate, the 18-21 year olds willing to join the military may be dependent on the economic health of the nation's private sector, which in turn may depend partly on military expenditures. It then becomes difficult to decide whether to make the available 18-21 year old population an exogenous
variable to be provided when needed by the model, or expand the model to include the economy as a whole and split the 18-21 year old population into two endogenous variables -- one of those willing to join and the other of those unwilling, with the ratio dependent on economic health. In this example, the model expansion would be so major that the modeler would probably choose to make the labor pool variable exogenous, and provide a table for the labor pool as a function of time.

One of the modeler's most difficult tasks is the determination of the system boundary. Usually one starts with the variables known to be internal, and then forms linkages until potential system boundaries are reached. Often the final boundary is not determined until late in the model building stage. The compromises made in determining which variables to include and which to exclude will often be less than obvious. Some arbitrary decisions will need to be made. Incidentally, these decisions should be documented, as model justification will be facilitated later if a time track of model decisions is recorded.

2.8 Model Sectors

The model boundary contains endogenous variables linked to the environment through parameters and exogenous variables. It isolates the system under study. Within the boundary, most models contain subsystems or "sectors" linked to variables in other sectors. These linkages are similar to those linking the model variables to the environment, but in this case the linkage
is through system variables. Variables in a sector will change as the variables to which they are linked change.

For example, in a small supplier's inventory problem, one sector might be the inventory sector, comprised of the inventory, orders, average sales, and desired inventory. Sales and delivery delay are exogenous. Orders and sales affect inventory, desired inventory depends on average sales, and orders depend on the difference between desired inventory and inventory. Figure 2.8.1 applies.

![Inventory Model Diagram](image)

**Figure 2.8.1: Inventory Model. Sales and Price Exogenous**

However, if the supplier holds a large share of the market, thereby being able to affect price, then the modeler might make sales an endogenous variable by linking sales to price and adding a pricing sector. Figure 2.8.2 applies.
Sales would be inversely related to price, another new endogenous variable. Sales and price now form a pricing sector, price being affected by the imbalance between desired and actual inventories.

This example may clarify the difference between endogenous system variables, and exogenous variables. With sales an exogenous variable, sales will vary with time, but the sales profile over time will be identical, no matter how the system is controlled. On the other hand, if sales depend on price, and price is endogenous, then the sales profile over time will be different for different ordering and pricing policies.

Clearly the latter problem is more complex, and difficult to solve analytically. Yet it is easy to analyze using a stock/flow approach. This highlights why system dynamics is an appropriate method for use by the management scientist.